HW 4

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Math 1A: Sections 2.3 - 2.5

Section 2.3

30. $\lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4}$

$= \lim_{x\to -4} \frac{(\sqrt{x^2+9}-5)(\sqrt{x^2+9}+5)}{(x+4)(\sqrt{x^2+9}+5)}$

$= \lim_{x\to -4} \frac{x^2+9-25}{(x+4)(\sqrt{x^2+9}+5)}$

$= \lim_{x\to -4} \frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)}$

$= \lim_{x\to -4} \frac{(x+4)(x-4)}{(x+4)(\sqrt{x^2+9}+5)}$

$= \lim_{x\to -4} \frac{x-4}{\sqrt{x^2+9}+5}$

$= \frac{-8}{10}$

$\boxed{= -\frac{4}{5}}$ ■

32. $\lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

$= \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$

$= \lim_{h\to 0} \frac{x^2-(x+h)^2}{x^2(x+h)^2 h}$

$= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2 h}$

$= \lim_{h\to 0} \frac{-2xh-h^2}{x^2(x+h)^2 h}$

$= \lim_{h\to 0} \frac{-h(2x-h)}{x^2(x+h)^2 h}$

$= \lim_{h\to 0} \frac{-2x+h}{x^2(x+h)^2}$

$= \frac{-2x}{x^2 x^2}$

$= \frac{-2x}{x^4}$

$\boxed{= -\frac{2}{x^3}}$ ■

Section 2.4

32. $\lim_{x\to 2} x^3 = 8$

Scratch:

If $0 < |x-2| < \delta$, then $|x^3-8| < \varepsilon$

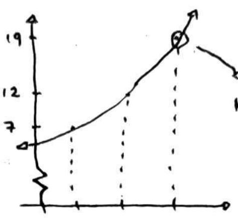
$|x^3-8| < \varepsilon$

$|(x-2)(x^2+2x+4)| < \varepsilon$

$|x-2||x^2+2x+4| < \varepsilon$

$|x-2||x^2+2x+4| < M|x-2| < \varepsilon$

32. cont'd.



$M = 19$

Assume:

$\delta \le 1$

$\delta < \frac{\varepsilon}{19}$

$|x-2||x^2+2x+4| < 19|x-2| < \varepsilon$; $|x-2| < \frac{\varepsilon}{19}$

$\therefore \delta = \min(1, \frac{\varepsilon}{19})$

Proof:

Given $\varepsilon > 0$, pick $\delta = \min(1, \frac{\varepsilon}{19})$. If $0 < |x-2| < \varepsilon$, then consider $|x^3-8| < \varepsilon$

$= |x-2||x^2+2x+4| < \delta|x^2+2x+4|$

$|x^2+2x+4|\delta < 19\delta$

$19\delta = 19(\frac{\varepsilon}{19}) = \varepsilon$

$\boxed{\text{Thus } |x^3-8| < \varepsilon \quad \text{Therefore } \lim_{x\to 2} x^3 = 8}$ ■

$|x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3$

$1 < x^2 < 9$ $\qquad$ $2 < 2x < 6$

$3 < x^2+2x < 15$

$7 < x^2+2x+4 < 19$

Section 2.5

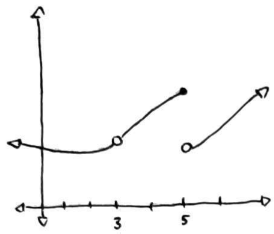
3. a. -4, undefind when $f(-4)$

-2, left and right not equal

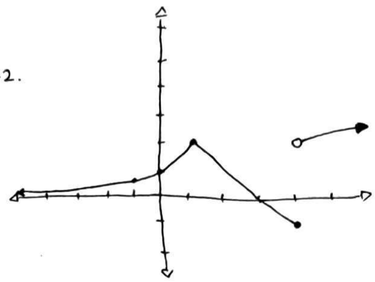
2, left and right not equal

4, left and right not equal

7.

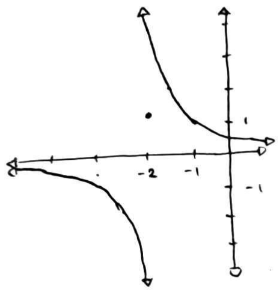


42.



$\boxed{4, \text{ left.}}$

18.



Function is discontinuous at $x = -2$ because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x) \neq f(x)$

23. $f(x) = \dfrac{x^2 - x - 2}{x-2}$

$= \dfrac{(x-2)(x+1)}{x-2}$

$\boxed{= x+1}$

36. $\lim_{x\to\pi} \sin(x + \sin x)$

let $f(x) = \sin x$
let $g(x) = x + \sin x$

$\lim_{x\to\pi} \sin(x + \sin x) = f(g(x))$

(continuity)

$= \sin(\pi + \sin\pi)$

$= \sin(\pi + 0)$

$= \sin\pi$

$\boxed{= 0}$

46.

$$f(x) = \begin{cases} \dfrac{x^4-1}{x-2} ; & x < 2 \\ ax^2 - bx + 3 ; & 2 \le x < 3 \\ 2x - a + b ; & x \ge 3 \end{cases}$$

$\lim_{x\to 2} \dfrac{x^4-1}{x-2} = \lim_{x\to 2} \dfrac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2 = 4$

$f(x)$ is continuous at $x < 3$ if: $4 = a(2)^2 + b(2) + 3$

$f(x)$ is continuous at $x \ge 2$ if: $a(3)^2 + b(3) + 3 = 2(3) - a + b$

$\therefore$ $\begin{cases} 4 = a(2)^2 - b(2) + 3 \\ a(3)^2 - b(3) + 3 = 2(3) - a + b \end{cases}$, solve for a & b.

$\cancel{4 = a(2)^2 + b(2) + 3}$
$\cancel{4 = 4a + 2b + 3}$

$4 = a(2)^2 - b(2) + 3$
$4 = 4a - 2b + 3$
$2b + 4 = 4a + 3$
$2b = 4a + 3 - 4$
$2b = 4a - 1$
$4b = 8a - 2$

substitute

$a(3)^2 - b(3) + 3 = 2(3) - a + b$
$9a - 3b + 3 = 6 - a + b$
$10a - 4b + 3 = 6$
$10a - 4b - 3 = 0$
$10a - (8a - 2) = 3$
$10a - 8a + 2 = 3$
$2a + 2 = 3$
$2a = 1$
$\boxed{a = \frac{1}{2}}$

substitute

$2b = 4a - 1$
$2b = 4\left(\frac{1}{2}\right) - 1$
$2b = 2 - 1$
$\boxed{b = \frac{1}{2}}$

52. Given that f is continuous on $[1,5]$ and the only solutions of the equation $f(x)=6$ are $x=1,4$, if $f(2)=8$ and $f(3)\le 6$, then $f(3)\ne 6$.

Because it is given that $f(x)$ will only be 6 when $x=1$ and $x=4$, therefore $f(3)<6$.

Knowing this, if $f(3)<6$, then f must cross 6 somewhere between 2 and 3 because of IVT.

$\boxed{\text{Therefore, because } x=1 \text{ and } x=4 \text{ will only result in } f(3)=6,\ f(3)>6, \text{ and IVT also works.}}$

53. $f(x) = x^4 + x - 3 = 0$

$f(1) = 1^4 + 1 - 3 = -1$

$f(2) = 2^4 + 2 - 3 = 15$

$\boxed{\text{To show that a root exists for } f(x) \text{ between } (1,2) \text{, we use IVT. Since } f(1)=-1 \text{ and } f(2)=15 \text{, and } f(1)\ne f(2) \text{, the root must exist between } (1,2) \text{ using IVT.}}$

69. $x^3 + 1 = x$

$x^3 - x + 1 = 0$

$f(x) = x^3 - x + 1$

$f(-2) = (-2)^3 - 2 + 1 = -9$

$f(1) = 1$

Using IVT, a root must exist. Therefore, there is a number that is 1 more than its cube.

2. $\lim_{x\to a} f(x) - \lim_{x\to a}\left(f(x) + \lim_{x\to a} g(x)\right)$

$\lim_{x\to a}\left(f(x) - (f(x) + g(x))\right)$

$\lim_{x\to a}(-g(x))$, if negative limit exists, positive should also

$-\lim_{x\to a}(g(x))$ ✓

$\boxed{\text{True.}}$

1. $\boxed{\text{False.}}$

Counter example:

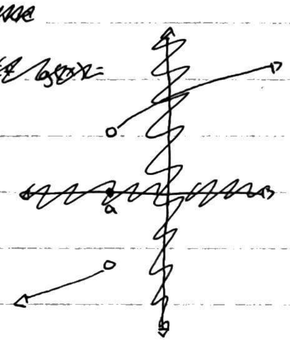
$g(x) = \begin{cases} \frac{x^2+4x+4}{x+2} ; & x \ne -2 \\ 1 ; & x=-2 \end{cases}$

$f(x) = 1$

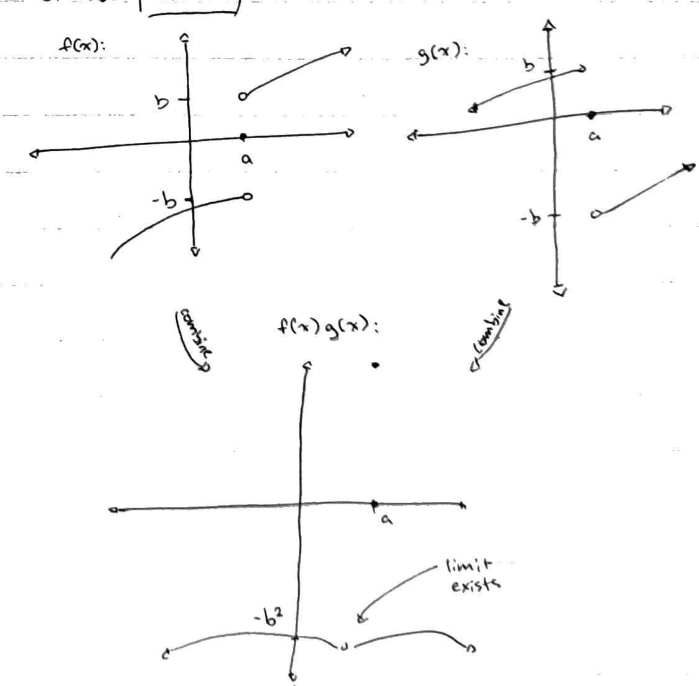
$\therefore \lim_{x\to -2} \frac{f(x)}{g(x)}$

$\boxed{= -\frac{1}{2}}$ ← Exists!

2.

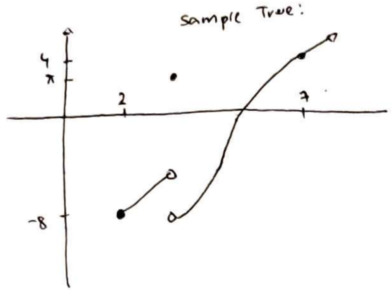


3. $\boxed{\text{False:}}$

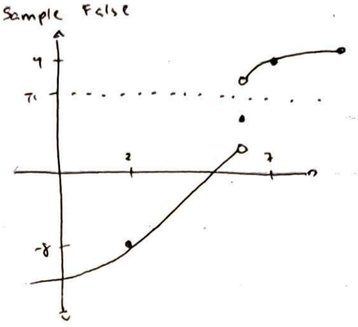


4. False. (Function needs to be continuous for this to be true, according to IVT.)

Sample True:



Sample False



Proves False.