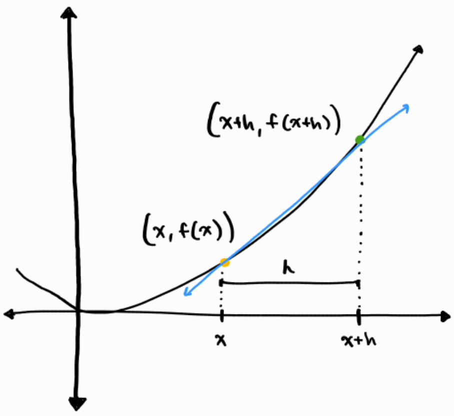
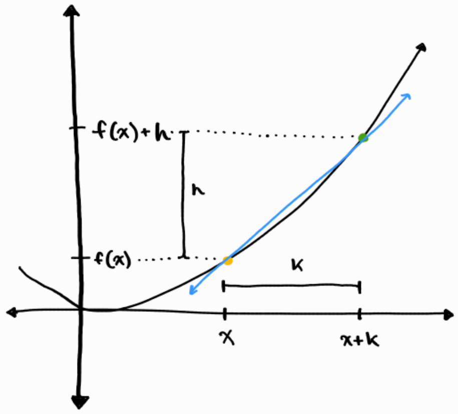
# Take Home Quiz 2

## AP Calc BC

1.




2.




3. $k = f^{-1}\left(f(x)+h\right) - x$

Given $f(x+k) = f(x) + h$

$\therefore f^{-1}\left(f(x)+h\right) = x+k$, by definition of an inverse

$k = (x+k) - x$

$\boxed{k = k}$ ✓ ∎

4. $\lim\limits_{h\to 0} \dfrac{f(x+k) - f(x)}{x+k-x}$

$= \lim\limits_{h\to 0} \dfrac{f(x) + h - f(x)}{k}$

$\boxed{= \lim\limits_{h\to 0} \dfrac{h}{f^{-1}\left(f(x)+h\right) - x}}$

5. a. $f(x) = \sqrt{x}$

$x = \sqrt{y} \rightarrow y = x^2$

$f^{-1}(x) = x^2 \quad x \in [0, \infty)$

$\lim\limits_{h\to 0} \dfrac{h}{f^{-1}\left(f(x)+h\right) - x}$

$= \lim\limits_{h\to 0} \dfrac{h}{f^{-1}\left(\sqrt{x}+h\right) - x}$

$= \lim\limits_{h\to 0} \dfrac{h}{(\sqrt{x}+h)^2 - x}$

$= \lim\limits_{h\to 0} \dfrac{h}{x + 2h\sqrt{x} + h^2 - x}$

$= \lim\limits_{h\to 0} \dfrac{h}{2h\sqrt{x} + h^2}$

$= \lim\limits_{h\to 0} \dfrac{1}{2\sqrt{x} + h}$

$\boxed{= \dfrac{1}{2\sqrt{x}} \quad x \in [0, \infty)}$ ∎

b. $f(x) = \dfrac{1}{x}$

$x = \dfrac{1}{y} \rightarrow y = \dfrac{1}{x}$

$f^{-1}(x) = \dfrac{1}{x}$

$\lim\limits_{h\to 0} \dfrac{h}{f^{-1}\left(f(x)+h\right) - x}$

$= \lim\limits_{h\to 0} \dfrac{h}{f^{-1}\left(\frac{1}{x}+h\right) - x}$

$= \lim\limits_{h\to 0} \dfrac{h}{\frac{1}{1/x+h} - x}$

$\dfrac{1}{1/x+h} = \dfrac{x}{1+hx}$

$= \lim\limits_{h\to 0} \dfrac{h}{\frac{x}{1+hx} - x}$

$\dfrac{x}{1+hx} - x$

$= \dfrac{x - x(1+hx)}{1+hx}$

$= \dfrac{x - x - hx^2}{1+hx}$

$= \dfrac{-hx^2}{1+hx}$

$= \lim\limits_{h\to 0} \dfrac{h}{\frac{-hx^2}{1+hx}}$

$= \lim\limits_{h\to 0} -\dfrac{1+hx}{x^2}$

$\boxed{= -\dfrac{1}{x^2}}$ ∎

6. $f(x) = x^2$

$x = y^2 \rightarrow y = \pm\sqrt{x}$

$f^{-1}(x) = \pm\sqrt{x}$

$$= \lim_{h\to 0} \frac{h}{f^{-1}(f(x)+h) - x}$$

$$= \lim_{h\to 0} \frac{h}{f^{-1}(x^2+h) - x}$$

$$= \lim_{h\to 0} \frac{h}{\pm\sqrt{x^2+h} - x}$$

a. $\sqrt{x^2} = |x|$

Positive number always guaranteed due to nature of $x^2$

b. By definition of a inverse function:

$$f^{-1}(f(x)) = x$$

Since $f^{-1}(x) = \pm\sqrt{x}$, to satisfy the definition of a inverse, we can say when $x \geq 0$, $f^{-1}(x) = \sqrt{x}$. Likewise, when $x < 0$, $f^{-1}(x) = -\sqrt{x}$.

Applying this, we can say the derivative of $x^2$ is:

$$\begin{cases} \lim\limits_{h\to 0} \dfrac{h}{\sqrt{x^2+h} - x} & ;\ x \geq 0 \\ \lim\limits_{h\to 0} \dfrac{h}{-\sqrt{x^2+h} - x} & ;\ x < 0 \end{cases}$$

$$= \begin{cases} \lim\limits_{h\to 0} \left[\dfrac{h}{\sqrt{x^2+h} - x} \cdot \dfrac{\sqrt{x^2+h} + x}{\sqrt{x^2+h} + x}\right] & ;\ x \geq 0 \\ \lim\limits_{h\to 0} \left[\dfrac{h}{-\sqrt{x^2+h} - x} \cdot \dfrac{\sqrt{x^2+h} - x}{\sqrt{x^2+h} - x}\right] & ;\ x < 0 \end{cases}$$

$$= \begin{cases} \lim\limits_{h\to 0} \dfrac{h(\sqrt{x^2+h} + x)}{(x^2+h) - x^2} & ;\ x \geq 0 \\ \lim\limits_{h\to 0} -\dfrac{h(\sqrt{x^2+h} - x)}{(x^2+h) - x^2} & ;\ x < 0 \end{cases}$$

$$= \begin{cases} \lim\limits_{h\to 0} \sqrt{x^2+h} + x & ;\ x \geq 0 \\ \lim\limits_{h\to 0} -(\sqrt{x^2+h} - x) & ;\ x < 0 \end{cases}$$

$$= \begin{cases} \sqrt{x^2} + x & ;\ x \geq 0 \\ -(\sqrt{x^2} - x) & ;\ x < 0 \end{cases}$$

$$= \begin{cases} |x| + x & ;\ x \geq 0 \\ -(|x| - x) & ;\ x < 0 \end{cases}$$

$$= \begin{cases} x + x & ;\ x \geq 0 \\ -(-x - x) & ;\ x < 0 \end{cases}$$

$$= \begin{cases} 2x & ;\ x \geq 0 \\ 2x & ;\ x < 0 \end{cases}$$

$$\boxed{= 2x} \ \blacksquare$$

I would say the difficulty when considering a negative value of $x$ is realizing $f^{-1}(x) = -\sqrt{x}$. From there, it's just a matter of evaluating the limit.