Take Home Quiz 3

AP Calculus BC

1. $f(x) = 2x^2 - x$

$$f'(x) = \lim_{h\to 0} \frac{\left[2(x+h)^2 - (x+h)\right] - (2x^2 - x)}{h}$$

$$= \lim_{h\to 0} \frac{2(x^2+2xh+h^2) - x - h - 2x^2 + x}{h}$$

$$= \lim_{h\to 0} \frac{2x^2+4xh+2h^2-h-2x^2}{h}$$

$$= \lim_{h\to 0} \frac{4xh+2h^2-h}{h}$$

$$= \lim_{h\to 0} 4x+2h-1$$

$$\boxed{= 4x-1} \blacksquare$$

2.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(a)}{h}$$

$$f''(x) = \frac{d}{dx} f'(x)$$

$$= \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+2h)-f(x+h)}{h} - \frac{f(x+h)-f(x)}{h}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+2h)-f(x+h)-f(x+h)+f(x)}{h^2}$$

$$\boxed{= \lim_{h\to 0} \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}} \blacksquare$$

3. $f(x) = 2x^2 - x$

$$\lim_{h\to 0} \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$$

$$= \lim_{h\to 0} \frac{\left[2(x+2h)^2-(x+2h)\right] - 2\left[2(x+h)^2-(x+h)\right] + (2x^2-x)}{h^2}$$

$$2(x+2h)^2-(x+2h)$$
$$= 2(x^2+4xh+4h^2)-(x+2h)$$
$$= 2x^2+8xh+8h^2-x-2h$$

$$2\left[2(x+h)^2-(x+h)\right]$$
$$= 2\left[2(x^2+2xh+h^2)-(x+h)\right]$$
$$= 2\left[2x^2+4xh+2h^2-x-h\right]$$
$$= 4x^2+8xh+4h^2-2x-2h$$

$$= \lim_{h\to 0} \frac{(2x^2+8xh+8h^2-x-2h) - (4x^2+8xh+4h^2-2x-2h) + (2x^2-x)}{h^2}$$

$$= \lim_{h\to 0} \frac{(2x^2-4x^2+2x^2)+(8xh-8xh)+(8h^2-4h^2)+(-x+2x-x)+(-2h+2h)}{h^2}$$

$$= \lim_{h\to 0} \frac{4h^2}{h^2}$$

$$\boxed{= 4} \blacksquare$$

4. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

$$= \lim_{h\to 0} \frac{\left[a_0 + a_1(x+2h) + a_2(x+2h)^2 + a_3(x+2h)^3\right] - 2\left[a_0 + a_1(x+h) + a_2(x+h)^2 + a_3(x+h)^3\right] + \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3\right]}{h^2}$$

$a_1(x+2h)$
$= a_1 x + 2a_1 h$

$a_2(x+2h)^2$
$= a_2(x^2 + 4x + 4h^2)$
$= a_2 x^2 + 4a_2 x + 4a_2 h^2$

$a_3(x+2h)^3$
$= a_3(x^3 + 6x^2 h + 12xh^2 + 8h^3)$
$= a_3 x^3 + 6a_3 x^2 h + 12a_3 xh^2 + 8a_3 h^3$

$a_1(x+h)$
$= a_1 x + a_1 h$

$a_2(x+h)^2$
$= a_2(x^2 + 2xh + h^2)$
$= a_2 x^2 + 2a_2 xh + a_2 h^2$

$a_3(x+h)^3$
$= a_3(x^3 + 3x^2 h + 3xh^2 + h^3)$
$= a_3 x^3 + 3a_3 x^2 h + 3a_3 xh^2 + a_3 h^3$

$$= \lim_{h\to 0} \frac{a_0 + a_1 x + 2a_1 h + a_2 x^2 + 4a_2 x + 4a_2 h^2 + a_3 x^3 + 6a_3 x^2 h + 12a_3 xh^2 + 8a_3 h^3 - 2a_0 - 2a_1 x - 2a_1 h - 2a_2 x^2 - 4a_2 xh - 2a_2 h^2 - 2a_3 x^3 - 2a_3 x^2 h - 6a_3 xh^2 - 2a_3 h^3 + a_0 + a_1 x + a_2 x^2 + a_3 x^3}{h^2}$$

$(2a_0 - 2a_0) + (2a_1 x - 2a_1 x) + (2a_1 h - 2a_1 h) + (2a_2 x^2 - 2a_2 x^2) + (4a_2 xh - 4a_2 xh) + (4a_2 h^2 - 2a_2 h^2) + (2a_3 x^3 - 2a_3 x^3) + (6a_3 x^2 h - 6a_3 x^2 h) + (12a_3 xh^2 - 6a_3 xh^2) +$ ↲

↳ $(8a_3 h^3 - 2a_3 h^3)$

$$= \lim_{h\to 0} \frac{2a_2 h^2 + 6a_3 xh^2 + 6a_3 h^3}{h^2}$$

$$= \lim_{h\to 0} 2a_2 + 6a_3 x + 6a_3 h$$

$$\boxed{= 2a_2 + 6a_3 x}$$ ∎

5. a.

$$f''(x) = \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

$$f'''(x) = \lim_{h\to 0} \frac{f''(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+3h) - 2f(x+2h) + f(x+h)}{h^2} - \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}}{h}$$

$$\boxed{= \lim_{h\to 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3}}$$ ∎

b. Looks like pascal's triangle:

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        1
       1 1
      1 2 1
     1 3 3 1
    1 4 6 4 1
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$$\lim_{h\to 0} \frac{f(x+4h) - 4f(x+3h) + 6f(x+2h) - 4f(x+h) + f(x)}{h^4}$$

c. Pascal's triangle is based off binomial expansion, where:

$$(x+y)^2 = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

From this definition, we can say:

$$(x-y)^2 = \sum_{k=0}^{n} \underbrace{\binom{n}{k} (-1)^k}_{\text{coeffecient}} x^{n-k} y^k$$

Therefore, given the pattern from $f' \ldots f^4$, the limit definition of f^n is:

$$\boxed{f^n(x) = \lim_{h\to 0} \left[\frac{1}{h^n} \cdot \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x+kh) \right]}$$ ∎